Orthonormal Sets

Defn (Unit vector)

Let H be a H.S. If $x \in H$ s.t. $\|x\|=1$ i.e. $(x,x)=1$ then x is called a unit vector or a normal vector.

Defn Orthonormal set

A non-empty subset $\{e_i\}$ of a H.S. H is said to be an orthonormal set if

(i) $\|e_i\|=1 \;\forall i$

(ii) $i \neq j \Rightarrow e_i \perp e_j$ i.e. $(e_i, e_j)=0$

i.e. the subset contains mutually orthogonal unit vectors.

Ex l_2^n – H.S.

Consider $\{e_1, e_2, \ldots, e_n\} \subset l_2^n$

where e_i is the nth tuple with 1 in the ith place and 0's elsewhere.

This collection forms an orthonormal set in l_2^n.

Ex l_2 H.S

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Bessel's Inequality: (Thm)

If $\{e_i\}$ is an orthonormal set in a H.S. H, then $\sum |(x, e_i)|^2 \leq \|x\|^2$ for each vector 'x' in H.

Pf:-

[**Note:-** If $\{e_i\}$ is an orthonormal set in a H.S. H and if x is any vector in H, then the set $S = \{e_i : (x, e_i) \neq 0\}$ is either empty or countable]

Let $S = \{e_i : (x, e_i) \neq 0\}$.

Then S is either empty or countable.

If S is empty then we have —

$(x, e_i) = 0 \quad \forall i$

Then we define $\sum |(x, e_i)|^2$ as number '0' and then we have $0 \leq \|x\|^2$.

$\therefore$ if S is empty then we have —

$$\sum |(x, e_i)|^2 \leq \|x\|^2.$$

Now suppose S is non-empty.

Then S is either finite or countably infinite. If S is finite then we can write $S = \{e_1, e_2, \ldots, e_n\}$ for some +ve int 'n'. In this case, we define

$$\sum |(x, e_i)|^2 = \sum_{i=1}^{n} |(x, e_i)|^2 \leq \|x\|^2,$$

by Bessel's inequality for finite set.

Now suppose that S is countably infinite. (21)
Let the vectors in S be arranged in a definite order as:

$$S = \{e_1, e_2, \ldots, e_n, \ldots\}$$

In this case, we define–

$$\sum |(x, e_i)|^2 = \sum_{n=1}^{\infty} |(x, e_n)|^2 \rightarrow (1)$$

This sum will be well defined if we can show that the series $\sum_{n=1}^{\infty} |(x, e_n)|^2$ is convergent and its sum does not change by rearrangement of its terms.
By Bessel's in-equality for finite cases, we have–

$$\sum_{i=1}^{n} |(x, e_i)|^2 \leq \|x\|^2 \rightarrow (2)$$

Since this relation (2) is true for every +ve int 'n', $\therefore$ it must also be true in the limit. So we have–

$$\sum_{n=1}^{\infty} |(x, e_n)|^2 \leq \|x\|^2 \rightarrow (3)$$

$\therefore$ from (3), series $\sum_{n=1}^{\infty} |(x, e_n)|^2$ is convergent. Since all terms of this series are +ve, $\therefore$ it is absolutely convergent and so its sum will not change by any rearrangement of its terms (22)

$\therefore$ we are justified in defining

$$\sum |(x, e_i)|^2 = \sum_{n=1}^{\infty} |(x, e_n)|^2$$ and

from (3), we have

$$\sum |(x, e_i)|^2 \leq \|x\|^2$$

(Proved)